傅利葉級數

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若 $f(x)$ 爲一週期爲 2π 之 *pointwise smooth function*，則可表爲

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\big(A_n\cos(nx) + B_n\sin(nx)\big)$$

其中 A_0, A_n, B_n 分別爲

$$\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad \text{for } n = 1, 2, 3, \cdots \\
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad \text{for } n = 1, 2, 3, \cdots
\end{aligned}$$

1 第一題

$$
\begin{aligned}
f(x) &= x \\
A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x dx \\
&= 0 \quad \text{因 } x \text{ 爲奇函數}
\end{aligned}
$$

$$
\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx \\
&= 0 \quad \text{因 } x\cos(nx) \text{ 爲奇函數}
\end{aligned}
$$

$$
\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx \quad \text{因 } x\sin(nx) \text{ 爲偶函數} \\
&= \frac{2}{\pi}\Big(\big(-\frac{x}{n}\cos(nx)\big)\Big|_0^\pi - \big(\int_0^\pi -\frac{1}{n}\cos(nx)dx\big)\Big) \quad \text{分部積分} \\
&= \frac{2}{\pi}\Big(\big(-\frac{\pi}{n}\cos(n\pi)\big) + \frac{1}{n^2}\sin(n\pi)\Big) \\
&= \frac{2}{\pi}\big(-\frac{\pi\cos(n\pi)}{n}\big) \quad \text{因 } sin \text{ 部份爲 } 0 \\
&= \frac{-2\cos(n\pi)}{n}
\end{aligned}
$$

$$
f(x) = \sum_{n=1}^{\infty} \frac{-2\cos(n\pi)}{n}\sin(nx)
$$

2 第二題

$$f(x) = |x|$$

$$\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}|x|dx \quad \text{因 } |x| \text{ 爲偶函數} \\
&= \frac{2}{\pi}\int_{0}^{\pi} xdx \quad \text{因範圍爲 } [0,\pi] \\
&= \frac{2}{\pi}(\frac{1}{2}\pi^2) \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}|x|\cos(nx)dx \quad \text{因 } |x|\cos(nx) \text{ 爲偶函數} \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)dx \quad \text{因範圍爲 } [0,\pi] \\
&= \frac{2}{\pi}\Big((\frac{x}{n}\sin(nx))|_0^\pi - \int_0^\pi \frac{1}{n}\sin(nx)dx\Big) \quad \text{分部積分} \\
&= \frac{2}{\pi}\Big((\frac{\pi}{n}\sin(n\pi)) - (-\frac{\cos(n\pi)+1}{n^2})\Big) \\
&= \frac{2}{\pi}(\frac{\cos(n\pi)-1}{n^2}) \quad \text{因 } \sin \text{ 部分全爲 } 0 \\
&= \frac{2\cos(n\pi)-2}{n^2\pi}
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx)dx \\
&= 0 \quad \text{因 } |x|\sin(nx) \text{ 爲奇函數}
\end{aligned}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2\cos(n\pi)-2}{n^2\pi}\cos(nx)$$

3 第三題

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

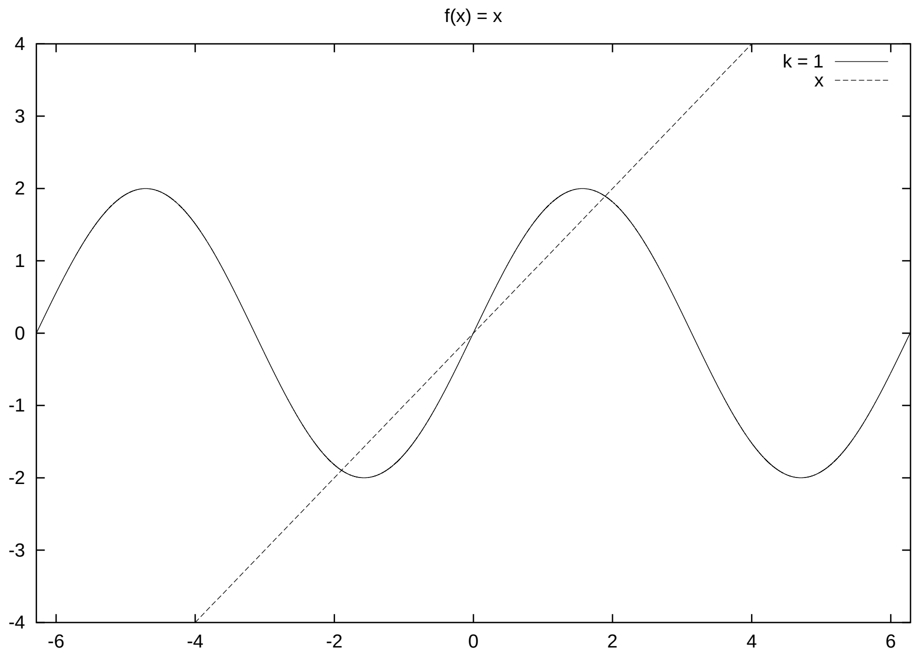
$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= 0 \quad \text{因 } f(x) \text{ 爲奇函數} \end{aligned}$$

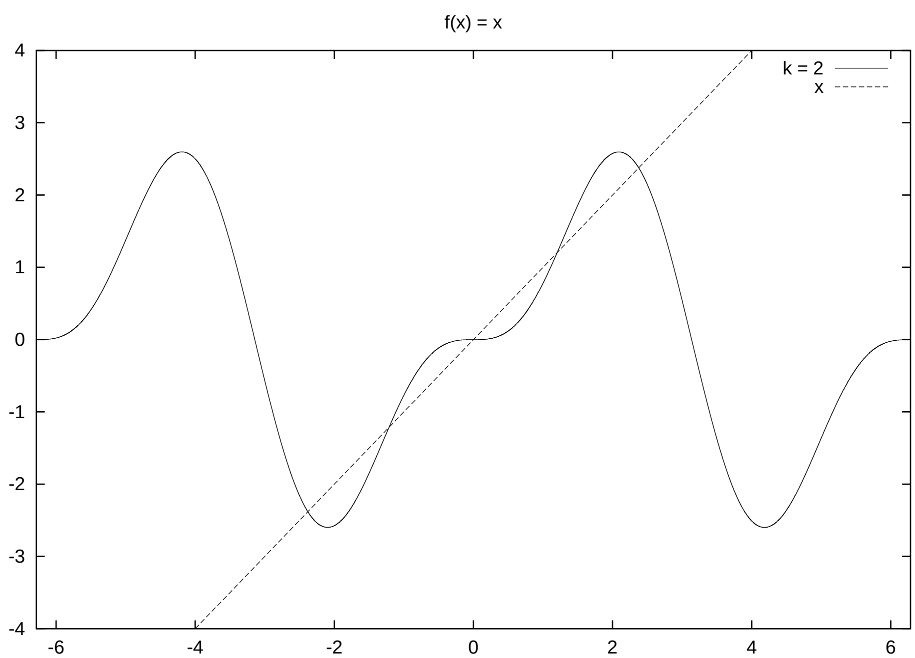
$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \\ &= 0 \quad \text{因 } f(x)cos(nx) \text{ 爲奇函數} \end{aligned}$$

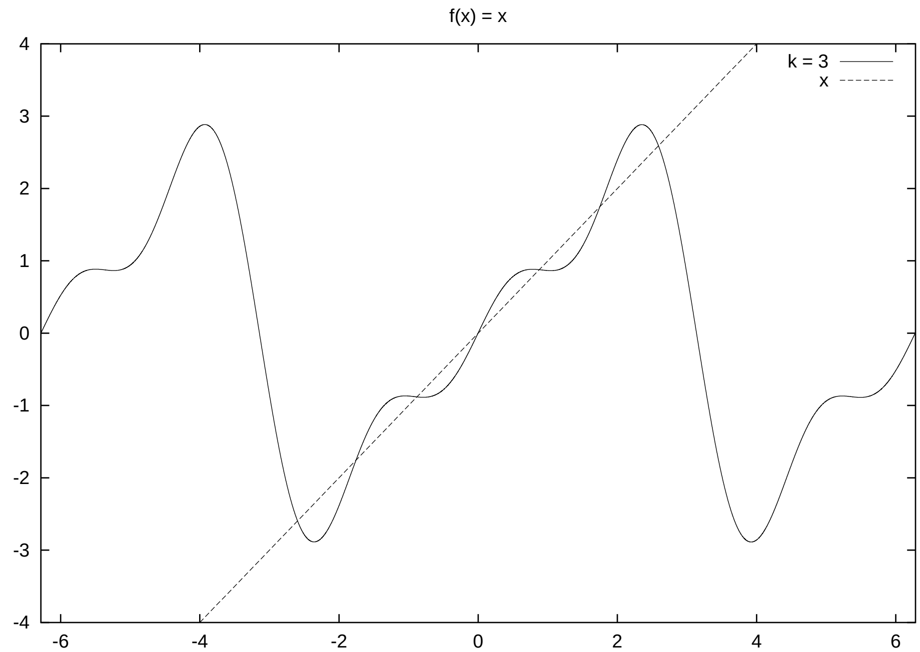
$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)sin(nx)dx \quad \text{因 } f(x)sin(nx) \text{ 爲偶函數} \\ &= \frac{2}{\pi}\int_{0}^{\pi} sin(nx)dx \quad \text{因範圍爲 } [0,\pi] \\ &= \frac{2}{\pi}(-\frac{\cos(n\pi)-1}{n}) \\ &= \frac{2-2\cos(n\pi)}{n\pi} \end{aligned}$$

$$f(x) = \sum_{1}^{\infty} \frac{2-2\cos(n\pi)}{n\pi} sin(nx)$$

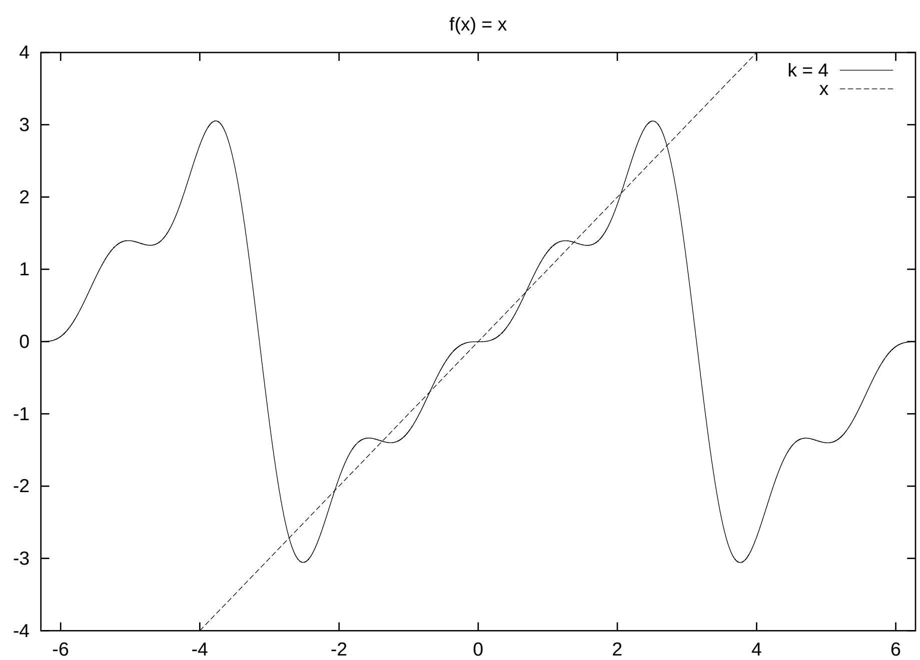
4 第四題


f(x) = x
k = 1
x

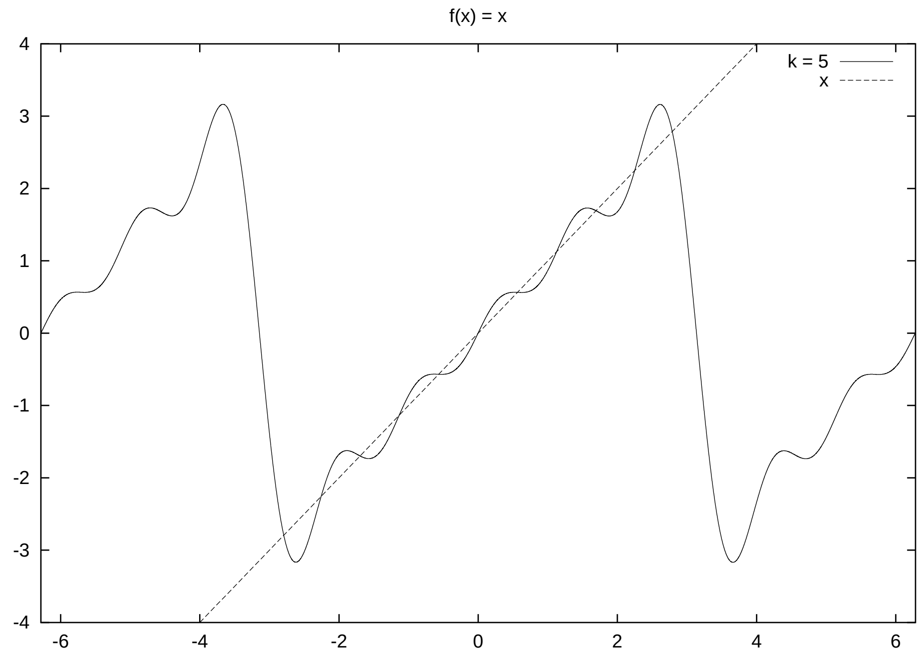

f(x) = x
k = 2
x

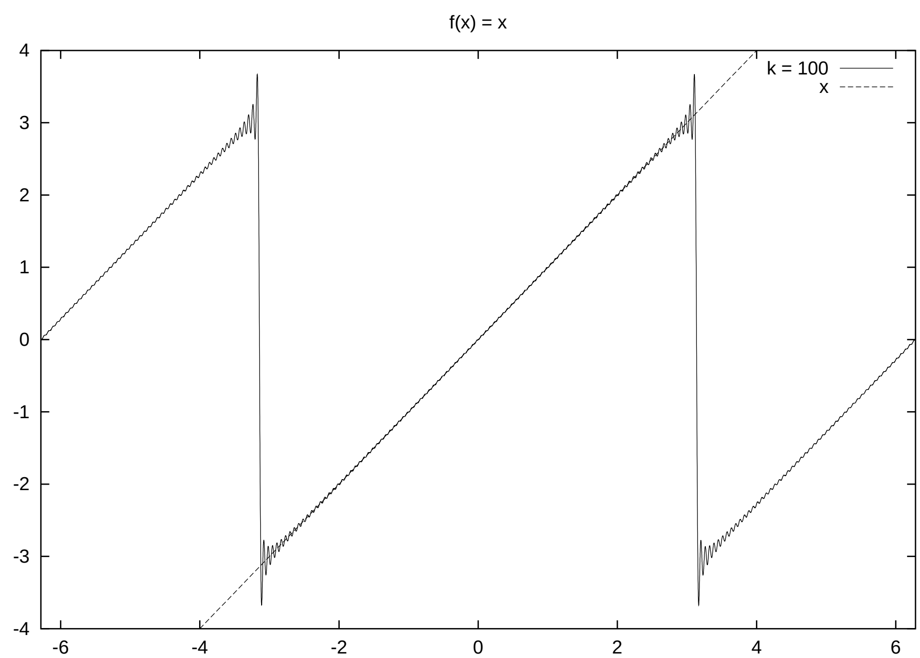


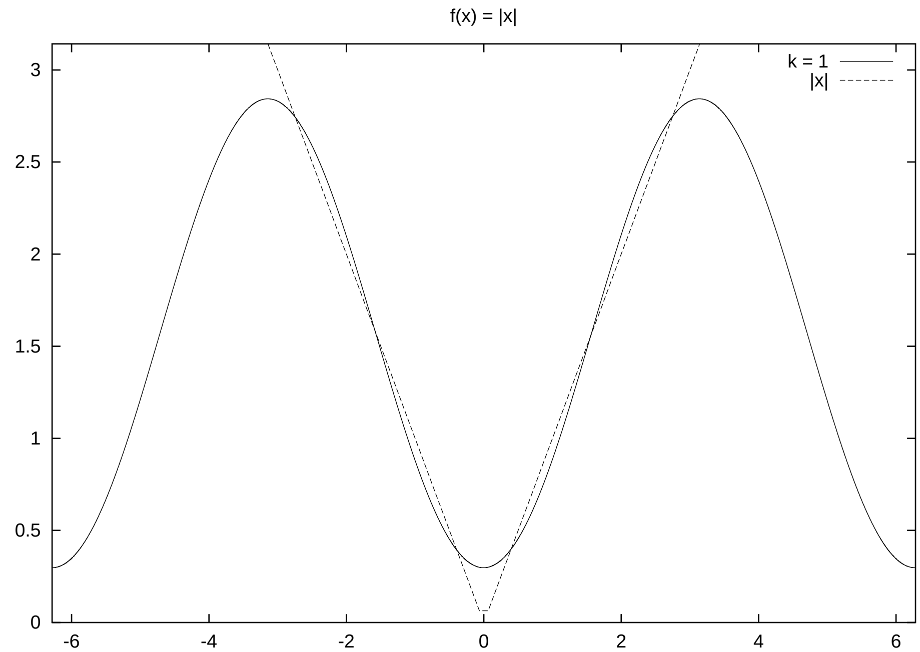
f(x) = x
k = 3
x
4
3
2
1
0
-1
-2
-3
-4
-6
-4
-2
0
2
4
6

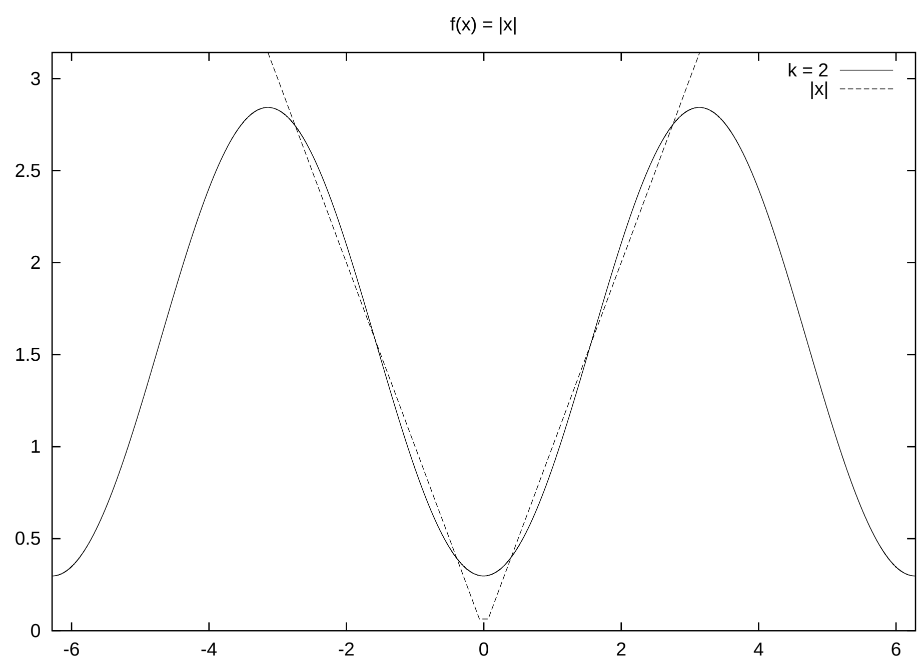


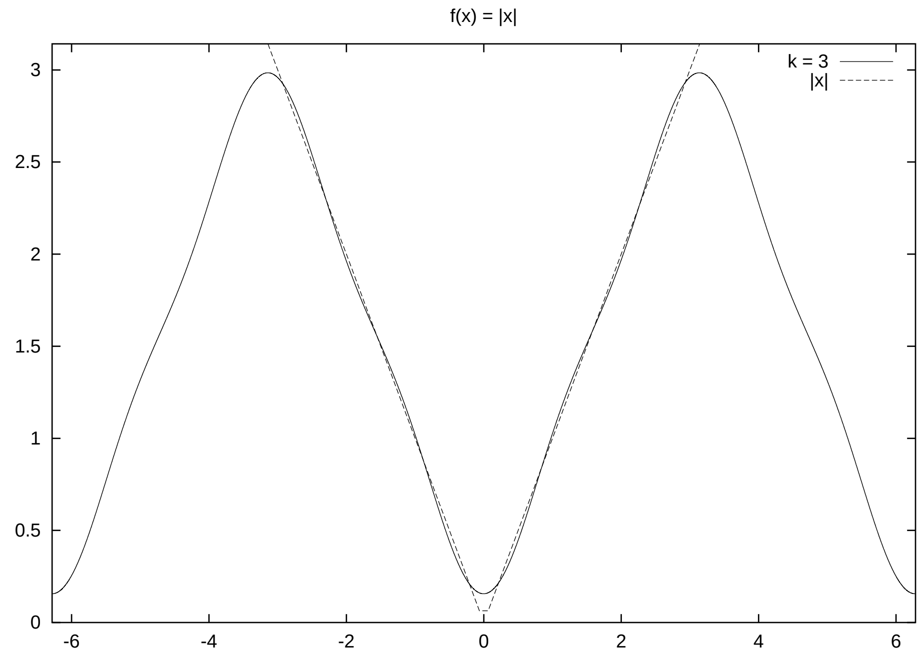
f(x) = x
k = 4
x
4
3
2
1
0
-1
-2
-3
-4
-6
-4
-2
0
2
4
6

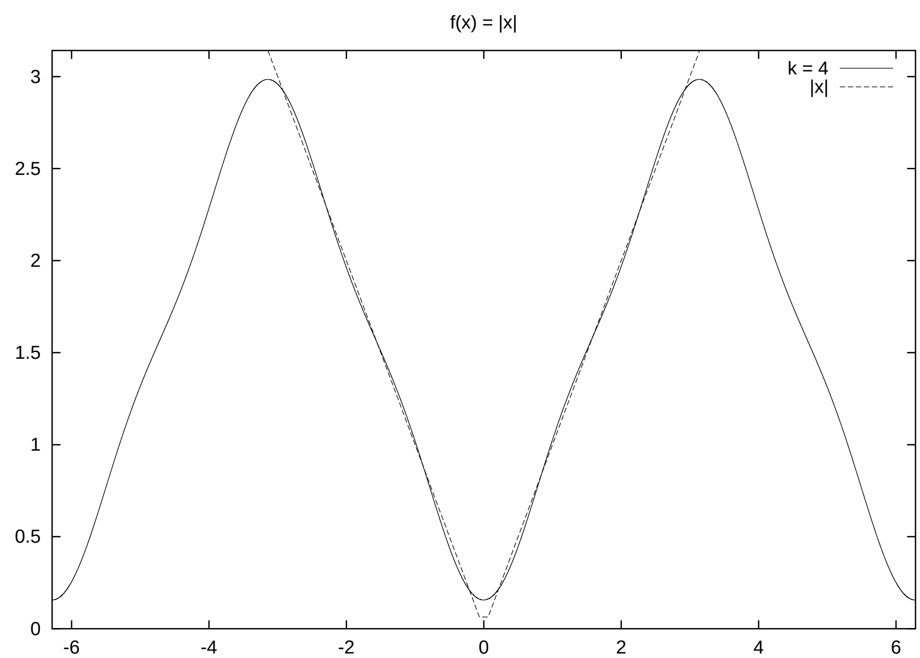

f(x) = x
k = 5
x
4
3
2
1
0
-1
-2
-3
-4
-6
-4
-2
0
2
4
6

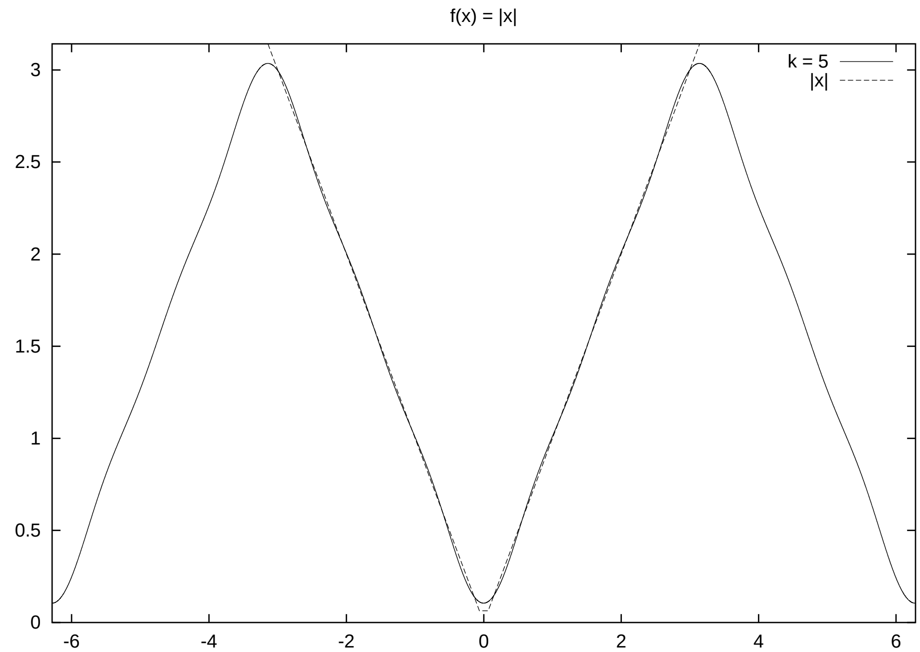

f(x) = x
k = 100
x
4
3
2
1
0
-1
-2
-3
-4
-6
-4
-2
0
2
4
6

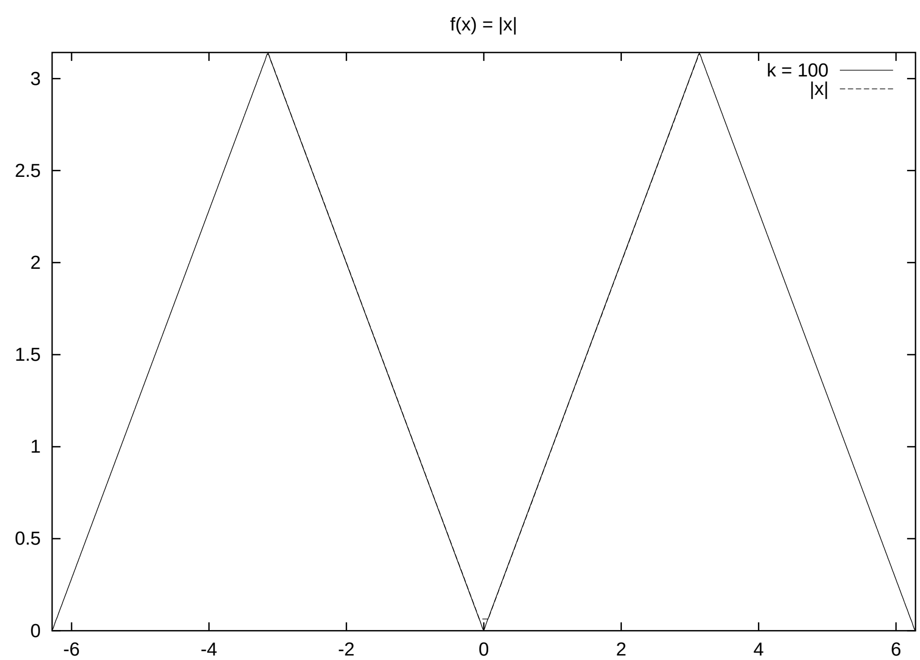

f(x) = |x|
k = 1
|x|
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6

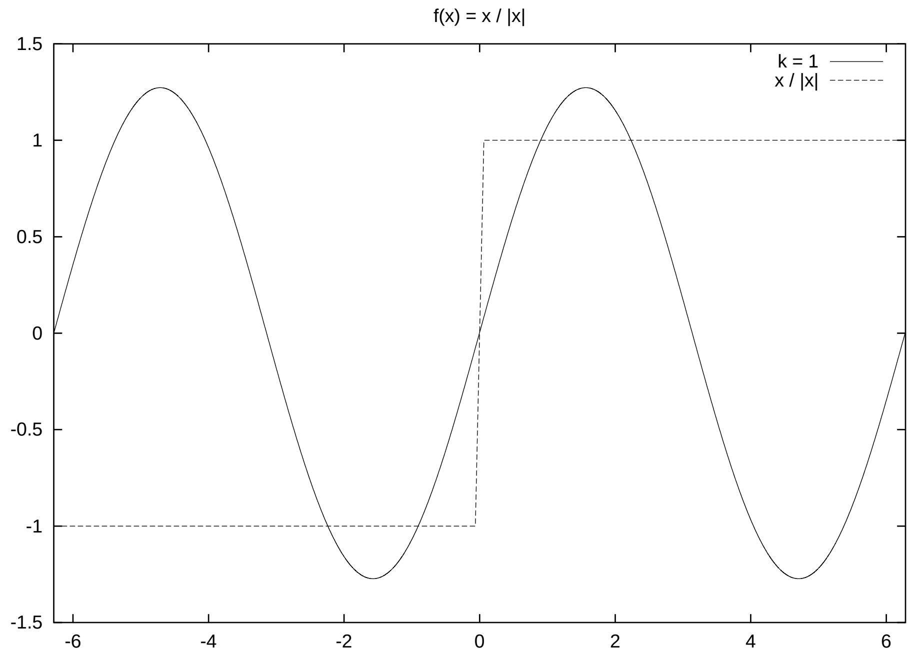

f(x) = |x|
k = 2
|x|
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6

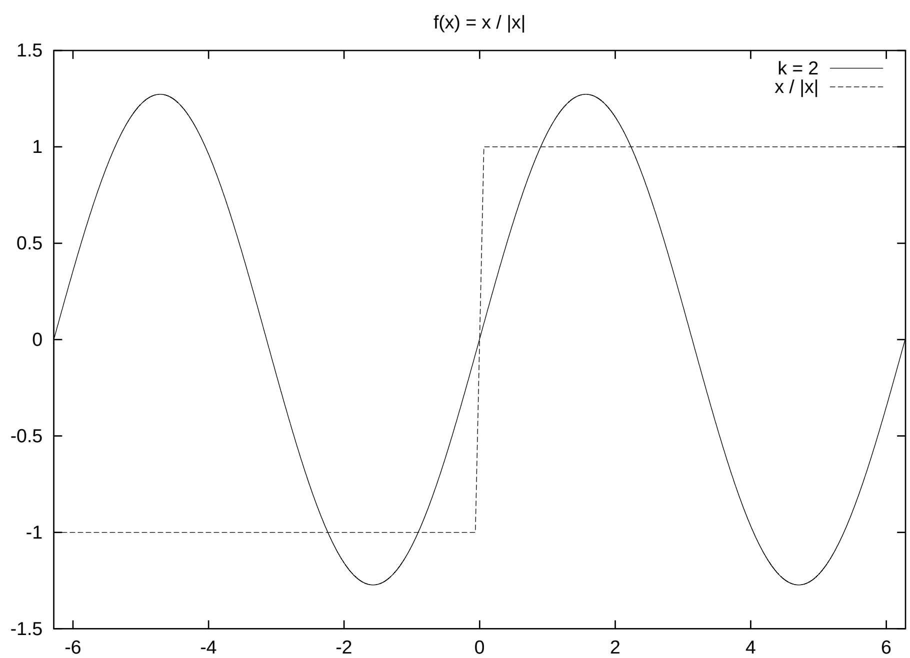

f(x) = |x|
k = 3
|x|
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6

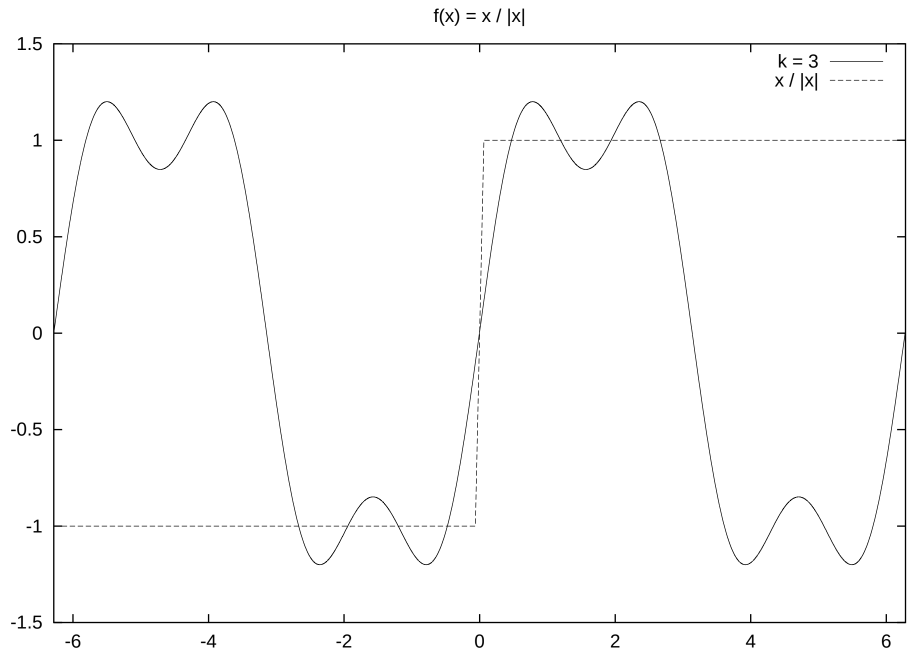

f(x) = |x|
k = 4
|x|
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6

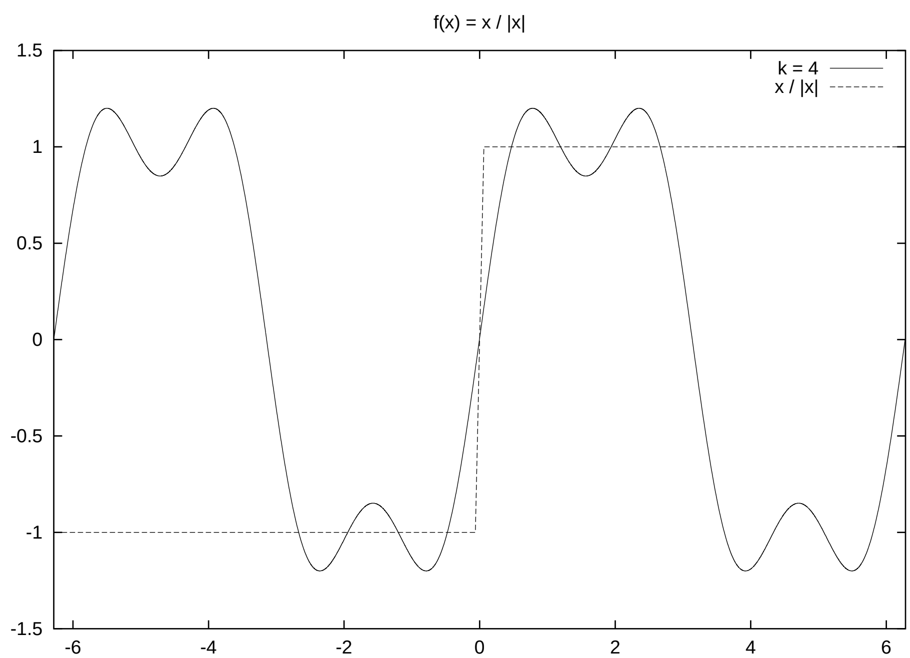

f(x) = |x|
k = 5
|x|
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6

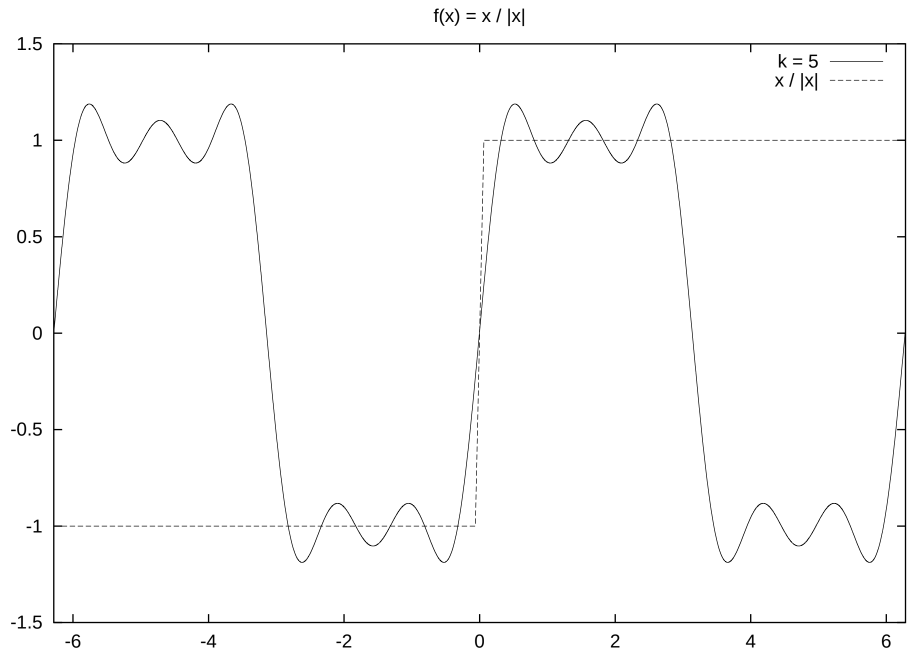

f(x) = |x|
k = 100
|x|
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6

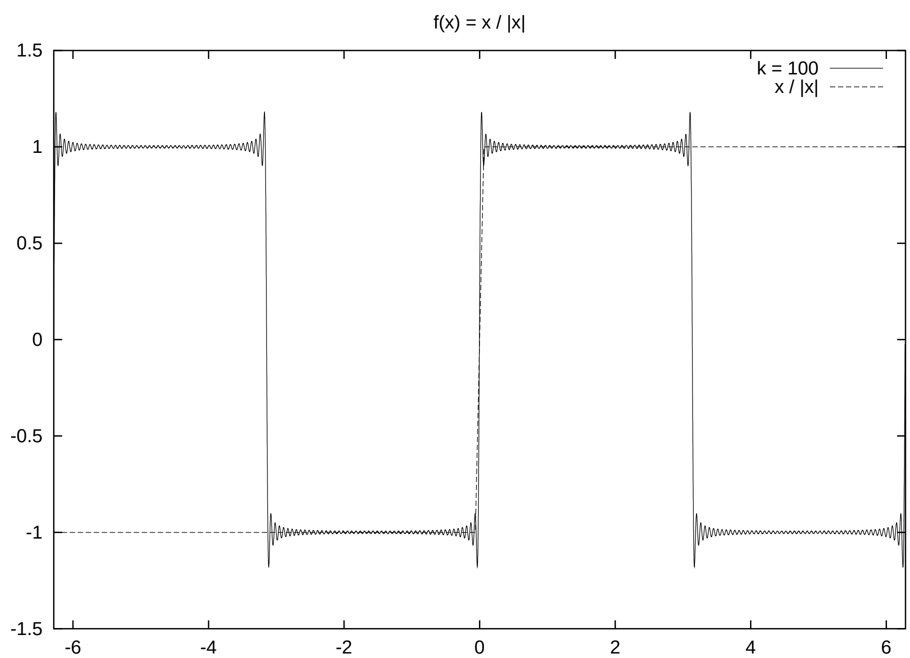

f(x) = x / |x|
k = 1
x / |x|
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6


f(x) = x / |x|
k = 2
x / |x|
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6


f(x) = x / |x|
k = 3
x / |x|
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6


f(x) = x / |x|
k = 4
x / |x|
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6


f(x) = x / |x|
k = 5
x / |x|
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6


f(x) = x / |x|
k = 100
x / |x|
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6

5 參考資料

- James S. Walker, Fourier Analysis
- Michael Downes, Short Math Guide for LaTeX

6 使用工具

- FreeBSD, http://www.freebsd.org/
- TeX, http://www.tug.org/
- LaTeX, http://www.latex-project.org/
- CJK, amsmath, graphicx, figure packages
- C Programming Language
- Gnuplot, http://www.gnuplot.info/
- Dvips, http://www.radicaleye.com/dvips.html
- Ghostscript, http://www.gnu.org/software/ghostscript/ghostscript.html
- Ghostview, http://www.gnu.org/software/ghostview/ghostview.html